Measuring Industry Contributions to Labour Productivity Change: A New Formula in a Chained Fisher Index Framework

Marshall Reinsdorf
International Monetary Fund[1]

ABSTRACT
Canada and the United States use Fisher indexes in their input-output accounts. Existing methods for decomposing aggregate labour productivity growth into industry contributions in a Fisher index framework either leave some productivity growth unaccounted for or are poorly suited for answering relevant questions about the industry sources of productivity growth. This article derives formulas for analyzing industry contributions to productivity change that add up exactly to the aggregate change in productivity and that have useful economic interpretations. These formulas show that the manufacturing sector made a positive contribution to productivity growth in the Canada in 2000-2010 and in the United States in 1998-2012, whereas the widely used GEAD formula implies that manufacturing made a negative contribution. Methods that can be used to decompose chained Laspeyres measures of productivity growth are also developed. These methods would be applicable in countries other than Canada and the United States.

ALTHOUGH THE EFFECTS ON wages and living standards of productivity growth tend to be broad-based, productivity gains tend to be concentrated in certain industries. An analysis of sector or industry contributions to aggregate productivity growth rates is therefore an important part of understanding an economy's productivity performance.

Productivity is always measured by comparing outputs to inputs, but two approaches are possible for defining inputs. First, labour productivity is generally measured using a simple sum of hours of labour inputs. Second, total factor productivity (TFP, also known as multifactor productivity or MFP) is measured with a quantity index of all inputs used in production. This article will focus on methods for analyzing sources of growth in labour productivity. For an economy that does not have large flows of cross-border investment income or large changes in the labour force participation rate, the standard of living ultimately

1 The author is Senior Economist in the Statistics Department at the International Monetary Fund. This article is a revised version of a paper presented at a session on productivity issues organized by the Centre for the Study of Living Standards (CSLS) at the annual meeting of the Canadian Economics Association, May 30-June 1, 2014, Simon Fraser University, Vancouver, British Columbia. At the time, the author was Chief of the National Accounts Research Group at the US Bureau of Economic Analysis. The author is grateful to Andrew Sharpe, Matthew Calver, Jianmin Tang, Doug May and an anonymous referee for helpful comments. The views expressed are those of the author and should not be attributed to the IMF, its managers or Executive Directors. Email: mreinsdorf@imf.org.

depends on labour productivity.[2] The growth of aggregate total factor productivity (TFP) is a critical determinant of labour productivity growth over the long run so an appendix contains a discussion of an additive decomposition of aggregate TFP growth.

The article is organized as follows. In section one, the challenges in the measurement of the contribution of each sector or industry to overall productivity change are identified as the effect of movement of labour between sectors and as the non-additivity introduced by the Fisher index formula and by chaining.[3] In the second section, the three main formulas in the literature on calculating industry contributions to overall productivity change are presented and their advantages and disadvantages are reviewed. The contrasting behaviour of these formulas is then illustrated with data from Canada. For example, according to the CSLS decomposition formula, the total contributions of mining and construction to Canada's productivity growth are negative and the total contribution of manufacturing is positive, but according to the "Generalized Exactly Additive Decomposition" or GEAD formula, the contributions of mining and construction are positive while the contribution of manufacturing is negative.

To overcome the disadvantages of the existing formulas, section three proposes a new formula for exactly decomposing productivity change in a Fisher framework. It then develops a simplified approximation to this formula that is easy to calculate. In an empirical example using data from the United States, the Fisher decomposition formula identifies manufacturing of computers and other durable goods as making a large positive contribution to US productivity, whereas the GEAD formula implies that they dragged down US productivity growth. Also, the GEAD formula identifies health care and the category that contains fast food restaurants as making positive contributions to US productivity growth, while the Fisher decomposition formula implies their contributions were negative.

Challenges in Decomposing Productivity Growth

The first consideration in designing a formula for calculating contributions of individual industries or sectors to aggregate labour productivity growth is that the contributions are supposed to add up exactly to the change in labour productivity at the aggregate level. The aggregate of interest may be all of GDP, the business sector excluding the real estate industry, or a segment of the business sector, such as private business.

Calculating additive contributions to aggregate productivity growth is not a simple problem because differences in inputs of physical, intangible, and human capital, and also differences in technology, cause industries to vary in their labour productivity levels. Differences in labour

2 Changes in prices of exports and imports also affect living standards, but their effects are usually more transitory than effects of productivity gains.

3 The properties of a measure of real output are affected by the choice of index formula and by whether the long-run indexes have a fixed base or are chained. If a fixed base approach is used for the long-run indexes, quantities from all the years are valued at a set of constant prices from an arbitrary base period, while if an annual chaining approach is used for the long-run indexes, the base period changes every year. In the Laspeyres index formula the initial period is the base period, while in the Paasche index formula the final period is the base period. Thus, in the Laspeyres volume index, the initial period quantities and the final period quantities are both valued at the prices of the initial period, while in the Paasche index, the initial period quantities and final period quantities are both valued at the prices of the final period. The Fisher index is the square root of the product of the Laspeyres index and the Paasche index, that is the geometric mean. The short-term indexes used for the annual links that are chained together may be calculated as Laspeyres, Paasche, or Fisher indexes.

productivity levels across industries mean that movement of labour between industries changes aggregate productivity, a phenomenon known as the "labour reallocation effect".[4]

Assigning a jointly produced effect to individual actors involves a certain amount of arbitrariness, so one option for handling the labour reallocation effect is to leave it out of the decomposition. Yet productivity analysts prefer to have a set of industry contributions that completely account for aggregate productivity growth with no residual. This article will argue that economically meaningful contributions to the reallocation effect can be calculated using one of the procedures that have been proposed to decompose the labour reallocation effect.

Another challenge arises in analyzing productivity change using official data from Canada or the United States. Labour productivity is a ratio whose numerator is an output volume measure that either has been constructed by deflating nominal output by a price index or, equivalently, by multiplying base period nominal output by a quantity index. The statistical agencies of Canada and the United States use chained Fisher indexes for this purpose. These indexes have many desirable properties, but Fisher indexes and chained indexes also have the inconvenient property of yielding output volume measures that are not additive.[5] Non-additivity means that a residual will generally exist between the sum of the real output of every industry (as measured by real value added) and real GDP. In contrast, real value added of every industry can be summed to obtain real GDP with no residual when the volume measures come from a "constant price" framework such as one that relies on direct (i.e. non-chained) Laspeyres or Paasche indexes.

The formulas that work well for calculating additive industry contributions to productivity change in a framework of constant price volume measures cease to be additive when applied in a Fisher or chained index framework. In effect, the discrepancy between the sum of the industry output volumes and the aggregate output volume that arises in a Fisher index or chained index framework will translate into a discrepancy between the sum of the productivity change contributions and the aggregate productivity change.

Main Formulas for Contributions to Labour Productivity Growth

Dumagan (2013) recently examined the characteristics of two widely used formulas for calculating contributions to labour productivity growth, which he terms, "the traditional decomposition" and "generalized exactly additive decomposition" or "GEAD." In addition to discussing these two formulas, de Avillez (2012) considered a modified version of the traditional decomposition that was developed by the Centre for the Study of Living Standards (CSLS) (Sharpe, 2010a and 2010b). This section will review these commonly used formulas, and also a variant of the GEAD introduced by Diewert (2013).

The "Traditional" Decomposition

When a single set of constant prices is used to calculate the output volume measures, the output volumes will be exactly additive. Additivity of the volume measures for industry output makes the problem of deriving an additive decomposition formula for labour pro-

4 Edward Denison identified the movement of labour from low-productivity level agriculture to high-productivity level manufacturing as a source of US labour productivity growth in the first half of the 20th century. Nordhaus (2002) calls the contribution to growth from labour reallocation "the Denison effect."

5 Diewert (1978) showed that a direct Fisher index (i.e. one that has not been chained) is approximately additive. We can therefore expect the residual caused by non-additivity to be small in the year immediately after the base year.

ductivity straightforward. The simplest formula for decomposing aggregate productivity change into industry contributions in a constant price framework has become known as "the traditional decomposition".

Let Z_{it} be the constant price measure of labour productivity in the arbitrary industry i and let Z_t be the corresponding measure of aggregate labour productivity. Also, let $l_{it} = L_{it}/L_t$ be the share of aggregate labour used by industry i, where L_{it} is the total of hours of labour used in industry i and L_t is the total over all industries of L_{it}. Then the l_{it} can be used as weights to express Z_t as an average of the Z_{it}:

$$Z_t = \Sigma_i l_{it} Z_{it} \qquad (1)$$

Expressing Z_0 in a similar way and breaking the $l_{it}Z_{it}$ - $l_{i0}Z_{i0}$ into a term for the change in productivity plus a term for the change in labour as a share of aggregate labour yields the traditional decomposition formula:

$$(Z_t - Z_0)/Z_0 = \Sigma_i([Z_{it}l_{it} - l_{i0}Z_{i0}]/Z_0)$$

$$= \Sigma_i[l_{i0}(Z_{it} - Z_{i0}) + Z_{i0}(l_{it} - l_{i0}) + (Z_{it} - Z_{i0})(l_{it} - l_{i0})]/Z_0$$

$$= (\Sigma_i((l_{i0}Z_{i0})/Z_0)((Z_{it}/Z_{i0}) - 1) + [(Z_{i0}/Z_0)(l_{it} - l_{i0}) + ((Z_{it} - Z_{i0})/Z_0)(l_{it} - l_{i0})])$$

$$= \Sigma_i(l_{i0}Z_{i0}/Z_0)g(Z_i) + [(Z_{i0}/Z_0)(l_{it} - l_{i0}) + g(Z_i)(l_{it} - l_{i0})]$$

$$= \Sigma_i \hat{c}_i^D + [(Z_{i0}/Z_0)(l_{it} - l_{i0}) + g(Z_i)(l_{it} - l_{i0})] \qquad (2)$$

Here the productivity growth rate of industry i, Z_{it}/Z_{i0}-1, has been denoted by $g(Z_i)$ and the direct contribution of within-industry productivity growth, $(l_{i0}Z_{i0}/Z_0)g(Z_i)$, has been denoted by $\hat{c}_i^D$. (The hat is to distinguish this term from the corresponding term of the GEAD formula discussed below, and the D stands for "direct".) The bracketed terms in equation (2) give contributions from labour reallocation, as is evident from their dependence on the change in each industry's share of employment. The term that depends on the base period level of productivity in each industry i will be called the static reallocation effect, while the term that depends on the growth rate of productivity will be called the dynamic reallocation effect.

The CSLS Decomposition

Assuming that all the Z_{it} are greater than 0, the reallocation effect terms in equation (2) imply that above average employment growth in an industry always contributes to productivity growth in a positive way. Equating fast employment growth to productivity growth in this way does not seem to have a sensible economic interpretation. The labour reallocation effect can, however, be decomposed into economically meaningful contributions by measuring each industry's productivity level as a deviation from the overall mean productivity level. Reinsdorf and Yuskavage (2010) offer a justification for such an approach, and Olley and Pakes (1995:1295) also provide an interpretation for this approach in their discussion of the covariance term of their additive expression for aggregate productivity growth. The modification of the traditional decomposition that uses deviations from means was called "the CSLS decomposition" by de Avillez (2012).

The impact on aggregate productivity of a movement of labour between industries depends on whether the productivity level is higher in the industries where the labour is redeployed than in the industries that it left. The CSLS decomposition uses the overall average level of productivity to account for the comparative productivity level of the industry that is receiving or releasing labour resources. In this decomposition, the reallocation effect component of an industry's contribution to aggregate productivity growth is positive if the industry has a below average productivity level and is releasing labour or if the industry has an above average productivity level and is absorbing labour.[6] As noted by Reinsdorf and Yuskavage (2010), an industry that releases labour can be viewed as placing that labour in a pool where it is available to any industry, and an industry that absorbs labour can be viewed as depleting the pool that is available to any industry. The economy's overall average level of productivity can therefore be treated as its opportunity cost of labour and used as a benchmark for measuring industry contributions to the labour reallocation effect.[7]

The CSLS decomposition has the same formula for the direct effect of within-industry productivity growth as the traditional decomposition, but its reallocation effect term is different. In the CSLS decomposition, the measures of productivity level and growth in the reallocation effect terms of equation (2) are re-expressed as deviations from means. In the static reallocation term, Z_{i0} - Z_0 is substituted for Z_{i0}, and in the dynamic reallocation term, the change in the mean productivity levels is subtracted from Z_{it} - Z_{i0}. These substitutions are possible because $\Sigma_i l_{it} - l_{i0} = 0$.

Let $\hat{c}_i^R$ be the reallocation effect term in which productivity levels and growth rates are expressed as deviations from means. Also, let the combined static and dynamic labour reallocation effect be:

$$\Sigma_i \hat{c}_i^R = \Sigma_i[(Z_{i0} - Z_0)/Z_0 + ((Z_{it} - Z_{i0}) - (Z_t - Z_0))/Z_0](l_{it} - l_{i0}) \qquad (3)$$

In equation (3) an industry that takes labour from other industries has a positive static contribution to aggregate productivity if its productivity level is above average, and it has a positive dynamic contribution to aggregate productivity if its productivity growth is above average. Conversely, an industry that releases labour to be employed by other industries has a positive static contribution to aggregate productivity if its productivity level is below average, and it has a positive dynamic contribution to aggregate productivity if its productivity growth is below average.

An appealing axiom for a formula for industry contributions to productivity change is that the difference between an industry's contribution and the average contribution for an industry of its relative size should be opposite in sign to the effect on the measure of aggregate productivity change of excluding that industry. In particular, if the contribution for the arbitrary industry *i* is greater than the benchmark "average" value of w_{i0}g(Z), where w_{i0} is the share of industry i in aggregate nominal output in period 0 and g(Z) is the aggregate productivity growth rate, then excluding that industry from the aggregate should have a downward effect on its growth rate. The CSLS formula has this desirable property.

Suppose that some industry, say industry *n*, matches the aggregate level of productivity in both time periods, so that $Z_{n0} = Z_0$ and $Z_{nt} = Z_t$. Then excluding this industry will not change aggregate productivity regardless of how its labour input changes. Consistent with this, if $l_{nt} \neq l_{n0}$ the CSLS formula estimate of this

6 The implicit assumption is that an industry's average productivity Z_{i0} is equal to or closely related to the productivity of the marginal labour that it releases or absorbs. An alternative to this assumption would be to assume that the marginal revenue product of labour in an industry is proportional to hourly compensation in that industry.

7 An alternative assumption would be that labour that is released goes into an industry with a productivity level of zero, and labour that is absorbed comes from this industry. This assumption implies the traditional decomposition and the "generalized exactly additive decomposition" (discussed below).

industry's contribution to aggregate productivity change will equal $w_{n0}g(Z)$.

Choice of Prices for the Volume Calculations in the Traditional or CSLS Decompositions

A weakness of the usual specification of both the traditional formula and the CSLS formula is that the selection of the reference period for the constant prices that are used to measure the Z_{i0} and the Z_{it} is left up to the discretion of the researcher. This makes the results depend in an arbitrary way on the researcher's choice of a base period for prices. Indeed, a researcher might even be able to manipulate the results by selecting a particular base period.

The relevant prices for measuring a volume change are those of the two periods between which the volume change is measured, so the reference period for prices should be one of these periods. The Laspeyres quantity index uses the prices of the initial period, while the Paasche quantity index uses the prices of the end period.

The Laspeyres quantity index is more convenient to use than the Paasche index. In the notation of equations (2) and (3), the Laspeyres quantity index takes its prices from period 0, making Z_{i0} equal to nominal output per hour. To calculate a Laspeyres volume measure for the arbitrary industry i, nominal output of industry i in time period t is deflated by a Paasche price index, denoted by P_{it}^{P}. This makes the prices used for the real output measure in Z_{it} also those of period 0.

Denote the contribution of within-industry productivity growth in industry i when period 0 is the base period for prices by c_i^D. Defining the base period for prices in this way allows $l_{i0}Z_{i0}/Z_0$ to be simplified to w_{i0}, so $c_i^D = w_{i0}g(Z_i)$. Substituting w_{i0} for $l_{i0}Z_{i0}/Z_0$ in all the terms of equation (2) yields a convenient expression for calculating the traditional decomposition:

$$
\begin{aligned}
(Z_t - Z_0)/Z_0 &= \Sigma_i w_{i0} g(Z_i) + w_{i0}(l_{it}/l_{i0} - 1) \\
&\quad + w_{i0} g(Z_i)(l_{it}/l_{i0} - 1) \\
&= \Sigma_i c_i^D + w_{i0}(l_{it}/l_{i0} - 1) \\
&\quad + w_{i0} g(Z_i)(l_{it}/l_{i0} - 1) \qquad (4)
\end{aligned}
$$

Making similar substitutions in the reallocation effect terms of equation (3), the reallocation effect part of the CSLS decomposition is:

$$
\begin{aligned}
\hat{c}_i^R &= l_{i0}[(Z_{i0} - Z_0)/Z_0 + Z_{i0}(Z_{it}/Z_{i0} - 1) \\
&\quad -(Z_t - Z_0))/Z_0](l_{it}/l_{i0} - 1) \\
&= [(w_{i0} - l_{i0}) + (w_{i0}(Z_{it}/Z_{i0} - 1) \\
&\quad - l_{i0}(Z_t/Z_0 - 1))]\ (l_{it}/l_{i0} - 1) \\
&= [w_{i0} - l_{i0} + w_{i0} g(Z_i) \\
&\quad - l_{i0} g(Z)](l_{it}/l_{i0} - 1) \qquad (5)
\end{aligned}
$$

The "Generalized Exactly Additive Decomposition"

A residual between the sum of the contributions and aggregate productivity growth will generally exist when a traditional or CSLS formula is applied in a Fisher or chained index framework. To eliminate this troublesome residual, researchers have sought a contributions formula that is exactly additive in all frameworks.

A solution to the problem of the residual was proposed by Tang and Wang (2004). Because the decomposition formula in Tang and Wang (2004) is applicable to superlative quantity index measures, to chained measures, and to Laspeyres volume measures, Dumagan (2013) terms it the "generalized exactly additive decomposition" or GEAD. This decomposition has been widely accepted in the literature and is often used in practice.

The GEAD formula normalizes the prices of individual industries by dividing each individual price by the deflator of the top-level aggregate. Let P_{it} denote the price index for the output (value added) of industry i, let F_t be the aggregate price index at time t, and let $p_{it} = P_{it}/F_t$ be the real price received by industry i at time t. Similarly, in period 0, $p_{i0} = P_{i0}/F_0$. In addition, let labour productivity in industry i be $X_{it} = (Y_{it}/P_{it})/L_{it}$. Then if Y_t is aggregate nominal value added in period t, aggregate labour productivity X_t is defined as $(Y_t/F_t)/L_t$. It can be written as:

$$X_t = \Sigma_i \, p_{it} \, l_{it} \, X_{it} \tag{6}$$

The change in labour productivity from period 0 to period t is:

$$g(X) \equiv \frac{X_t - X_0}{X_0}$$

$$= \frac{\Sigma_i p_{it} l_{it} X_{it}}{\Sigma_i p_{i0} l_{i0} X_{i0}} - 1 \tag{7}$$

Now let $w_{i0}= Y_{i0}/Y_0$ and note that $w_{i0}= p_{i0}l_{i0}x_{i0}/\Sigma_j \, p_{j0}l_{j0}x_{j0}$. Then:

$$g(X)= \Sigma_i \, w_{i0} \, [(p_{it}/p_{i0})(l_{it}/l_{i0})(X_{it}/X_{i0}) - 1]$$

$$= \Sigma_i \, w_{i0} \, [(p_{it}/p_{i0})(l_{it}/l_{i0})(1+g(X_i)) - 1]$$

$$= \Sigma_i \, w_{i0} \, [(p_{it}/p_{i0})(l_{it}/l_{i0}) - 1)(1+g(X_i)) + g(X_i)]$$

$$= \Sigma_i \, (w_{i0}/l_{i0})[(p_{it}/p_{i0})l_{it} - l_{i0}](1+g(X_i)) + w_{i0} \, g(X_i) \tag{8}$$

In the last line of equation (8), the term $(w_{i0}/l_{i0})[(p_{it}/p_{i0})l_{it} - l_{i0})](1+g(X_i))$ represents a reallocation effect c_i^R, which reflects both price growth and employment growth. The final term gives the direct contribution of within-industry labour productivity growth to aggregate productivity growth, $c_i^D = w_{i0} \, g(X_i)$. The formula for c_i^D shows that the GEAD uses the appropriate base period for the prices that underlie the volume measures used to calculate productivity.

The reallocation effect of equation (8) can be broken into a static reallocation effect and a dynamic reallocation effect. Defining x_{i0} as X_{i0}/X_0, the relative productivity of industry i in period 0, the total contribution from reallocation in the GEAD can be written as:

$$c_i^R = (w_{i0}/l_{i0})[(p_{it}/p_{i0})l_{it} - l_{i0}](1+g(X_i))$$

$$= (w_{i0}/p_{i0}l_{i0})(p_{it}l_{it} - p_{i0}l_{i0}) + (w_{i0}/p_{i0}l_{i0})(p_{it}l_{it} - p_{i0}l_{i0})g(X_i)$$

$$= x_{i0}(p_{it}l_{it} - p_{i0}l_{i0}) + x_{i0}(p_{it}l_{it} - p_{i0}l_{i0})g(X_i) \tag{9}$$

Reallocation of labour towards industry i occurs when $l_{it} > l_{i0}$. Under the assumption that $g(X_i) > -1$, the reallocation contribution will be positive if $p_{it}l_{it}/p_{i0}l_{i0} > 1$. If $p_{it} = p_{i0}$ then c_i^R is positive when the arbitrary industry i is a net recipient of reallocated labour and negative when it releases labour to be reallocated to other industries. Also, a sufficiently large increase in the price of the good that an industry produces (or fall in the prices of the goods that it uses as intermediate inputs) will cause p_{it}/p_{i0} to exceed l_{i0}/l_{it}, making $c_i^R > 0$. In the simple case of $l_{it} = l_{i0}$, c_i^R is positive when the real price that industry i receives for its output rises.

The Three-Component Version of the GEAD

Diewert (2013:5) has extended the approach of Tang and Wang (2004) to distinguish the price changes as a separate factor, in effect breaking apart the $p_{it}l_{it}$ effect of the standard GEAD formula. By treating all elements of the decomposition symmetrically, Diewert is able to express the total contribution of the arbitrary industry i to aggregate productivity growth as the sum of three similar terms. The terms of the three-component GEAD show the contributions of within-industry productivity growth, given the industry's price and labour growth; price growth, given the industry's productivity and labour growth; and labour share growth, given the industry's productivity and price growth, respectively:

$$c_i(\Delta X_i) = w_{i0} g(Xi)[1 + \tfrac{1}{2} g(p_i) + \tfrac{1}{2} g(l_i) + \tfrac{1}{3} g(p_i) g(l_i)] \quad (10)$$

$$c_i(\Delta p_i) = w_{i0} g(p_i)[1 + \tfrac{1}{2} g(X_i) + \tfrac{1}{2} g(l_i) + \tfrac{1}{3} g(X_i) g(l_i)] \quad (11)$$

$$c_i(\Delta l_i) = w_{i0} g(l_i)[1 + \tfrac{1}{2} g(X_i) + \tfrac{1}{2} g(p_i) + \tfrac{1}{3} g(X_i) g(p_i)] \quad (12)$$

Why a New Formula is Needed for the Fisher Volume Framework

The three-component version of the GEAD has an appealing symmetry in its treatment of each variable and both versions of the GEAD formula are versatile enough to produce exactly additive contributions regardless of the type of index used to create the output volume measures.

Nevertheless, the versatility of the GEAD comes at the cost of a lack of an appealing economic interpretation for the total contribution of an industry to the economy's productivity growth. The behavior of the total contribution is influenced by the treatment of changes in labour inputs and output prices in the term for the reallocation effect. Because of the treatment of labour input growth and output price growth, discrepancies are possible between the sign of the total contribution of the arbitrary industry *i* to aggregate productivity growth net of the size-adjusted average contribution of $w_{i0}\ g(X)$ and the sign of the change in aggregate productivity growth that would be caused by excluding industry *i* from the aggregate. Normally, excluding a high outlier from the sample results in a lower estimate of the mean, but in the case of the GEAD, excluding an industry whose total contribution is relatively large may paradoxically have a positive effect on aggregate productivity growth.[8]

This sort of paradox can arise because of the treatment of changes in labor inputs and output prices in the reallocation effect term of the GEAD. As in the traditional formula, rapid growth in employment in an industry is treatest as automatically adding to aggregate productivity growth. Yet, even though high employment growth in an industry can be a source of an increase in aggregate output if it means that the industry is providing jobs for people who would otherwise have been unemployed, raising aggregate output should not be confused with raising the average output per hour of those persons who are employed.[9]

Above average growth in an industry's output price has a similar effect on the reallocation effect term of the GEAD to above average growth in labour inputs. For example, suppose there is a disruption in the supply of an import that competes with a product produced by a domestic industry that causes the

8 Suppose that an industry with below average productivity growth has a large total contribution because rapid growth in its output price gives it a positive reallocation effect term. If the boundary of the aggregate whose productivity is being analyzed were redefined to exclude this industry, the new aggregate would have a lower aggregate price index and faster overall productivity growth.

9 To reflect the way the movement of labour into an industry with above average productivity raises overall productivity, the formula for c_i^R could be modified by replacing x_{i0} with x_{i0}-1. This substitution would, however, necessitate some further modification so that the contributions again add up to the correct total: $\Sigma_i (p_{it}l_{it} - p_{i0}l_{i0}) + \Sigma_i (p_{it}l_{it} - p_{i0}l_{i0}) g(X_i)$ may not equal 0.

domestic industry to enjoy a pricing windfall. The pricing windfall may result in the industry being counted as contributing positively to economy-wide productivity growth.

Nevertheless, treating an increase in the price that an industry receives for its output (or a decrease in the prices that it pays for intermediate inputs) as a positive contribution by the industry to aggregate productivity growth is inconsistent with the conceptual definition of productivity growth as an outward movement in the production possibility frontier caused by improvements in technology or in the organization of production.[10]

To avoid ambiguity, non-overlapping definitions of concepts are needed, so the definition of productivity growth must not be expanded so much that it encompasses direct effects of price movements. Prices do, of course, affect the weights in the quantity indexes that are used in the practical definition of productivity growth as the difference between the change in the quantity index for output and the change in the quantity index for inputs. However, these influences on weights are indirect effects, making the index more sensitive to some quantity changes and less sensitive to others. If every good has the same growth rate in its quantity, so that there is no substitution, prices will have no effect on the quantity index.

Including price effects with productivity gains as properly defined may cause true productivity developments to be masked by the price effects.[11] True productivity gains may be a *cause* of low or negative price changes because they result in cost savings that are passed on to buyers. When an industry experiences unusually high or usually low productivity growth, the impacts on real output and prices can be expected to tend to offset each other. The causality that generates an inverse relationship between industry productivity and industry output price can also run from prices to productivity. An exogenous increase in the price that an industry receives for its output may cause it to expand in a way that decreases its measured productivity because it operates under decreasing returns to scale; for example, mining industries tend to open or operate sites with lower grades of ore or more difficult conditions only when prices are high.

The three-component GEAD is more flexible in its treatment of price changes than the original GEAD, so it offers a partial solution for those analysts who want to exclude price changes from the contributions to productivity growth. If the price change term $c_i(\Delta p_i)$ is left out when adding up the total contribution, then only a small residual between the sum of the total contributions and the growth rate of aggregate productivity will be generated. Nevertheless, the property of exact additivity is sacrificed under this approach.

An Illustration using Data for the Canadian Business Sector

To illustrate the differences between the existing decomposition methods, Table 1 presents alternative decompositions of productivity growth of the Canadian business sector between 2000 and 2010 into contributions from two-digit NAICS sectors.[12] For the CSLS decomposition, the base year for prices is defined to match the base year for productivity growth, so the CSLS decomposition

10 In some contexts, a rise in real GDP caused by eliminating allocative inefficiencies might also be called a productivity gain, and in practice it would likely be measured as a productivity gain, but movements along the production possibility frontier to a more efficient allocation are conceptually distinct from outward shifts in the position of the frontier.

11 In the definition of productivity as real output divided by real inputs, prices have an indirect role as weights. Yet direct effects of price changes on revenue or input costs are excluded from the measures of change in real output or real inputs.

Table 1

Alternative Decompositions of Labour Productivity Growth in the Canadian Business Sector, 2000-2010

(simple averages of annual growth rates)

	Effect of Within-Sector Productivity Growth		Total Contributions		Excluding Price Term of Three-Component GEAD	
	CSLS and GEAD (1)	Three-Component GEAD (2)	CSLS (3)	GEAD and Three-Component GEAD (4)	Effect of Change in Real Price (5)	GEAD Total excluding Price Effect (6=4-5)
Business sector industries	0.82	0.65	0.76	0.80	-0.02	0.82
Agriculture, forestry, fishing and hunting	0.11	0.08	0.14	-0.06	-0.05	-0.01
Mining and oil and gas extraction	-0.21	-0.31	-0.06	0.28	0.29	-0.01
Utilities	0.00	0.00	0.01	-0.01	-0.02	0.01
Construction	0.01	0.01	-0.10	0.44	0.17	0.27
Manufacturing	0.23	0.18	0.07	-0.83	-0.33	-0.50
Wholesale trade	0.23	0.21	0.22	0.10	-0.05	0.15
Retail trade	0.17	0.17	0.15	0.14	-0.05	0.19
Transportation and warehousing	0.03	0.03	0.03	0.03	0.00	0.03
Information and cultural industries	0.10	0.10	0.10	0.06	-0.04	0.10
FIRE	0.04	0.05	0.16	0.20	-0.09	0.29
Professional, scientific & technical services	0.06	0.07	0.05	0.18	0.05	0.13
ASWMRS	0.01	0.01	-0.05	0.11	0.03	0.08
Arts, entertainment and recreation	0.00	0.00	-0.01	0.01	0.00	0.01
Accommodation and food services	0.02	0.02	0.03	0.02	0.01	0.01
Other private services	0.04	0.04	0.02	0.12	0.05	0.07

Source: CSLS calculations based on Statistics Canada Canadian Productivity Accounts.

Note: Output is defined as in business sector industries as total output in all two-digit NAICS sectors except for educational services, health care and social assistance, and public administration. Owner-occupied dwellings are excluded.

Note: ASWMRS is administrative and support, waste management and remediation services; FIRE is finance and insurance and real estate and rental and leasing.

gives the same numbers for the contributions of within-industry productivity growth as the standard GEAD formula. [13]

The contributions from within-sector improvements in labour productivity calculated from the CSLS and standard GEAD formulas slightly more than account for the aggregate productivity growth of the business sector of Canada, with of total of 0.82 percentage points (first column of numbers in Table 1). The man-

12 Data from Statistics Canada's Canadian Productivity Accounts were used to calculate these contributions. I am grateful to Andrew Sharpe and Matthew Calver from the Centre for the Study of Living Standards for assistance with these calculations.

13 Note that the CSLS contributions in Table 1 are not rescaled to add up to the aggregate productivity change, a step that is sometimes taken to deal with the non-additivity of the CSLS formula in a Fisher index framework. The discrepancy between the sum of the total contributions calculated with the CSLS formula, which is 0.76 percentage points, and the aggregate productivity growth rate of 0.80 per cent per year shows the size of the non-additivity problem when the CSLS or traditional formula is applied in a Fisher index framework.

ufacturing and wholesale trade sectors have the largest positive contributions from within-sector productivity gains, each 0.23 percentage points, while the mining and oil and gas extraction sector has a sizeable negative contribution, at –0.21 percentage points.

In the three-component GEAD, the contribution of within-sector productivity (equation (10)) includes separate terms for interactions with changes in prices and labour shares that are not included in the standard GEAD. The interactions make the mining sector's contribution more negative and make the manufacturing sector's contribution less positive; these changes in absolute magnitude reflect the fact that output prices were rising in the case of mining and oil and gas and falling in the case of manufacturing. Looking at the economy-wide totals, the three-component GEAD implies that about four-fifths of the business sector's productivity growth comes from within-sector labour productivity growth, about one-fifth comes from labour reallocation, and a small negative amount comes from the contributions of real price changes.

The total contributions of each sector to Canada's labour productivity growth are shown in the middle pair of columns of Table 1. The GEAD method gives a starkly different picture from the CSLS method. For example, according to the CSLS method, the agriculture, forestry and fishing sector made a large positive contribution to Canada's productivity growth, manufacturing made a modest positive contribution, and construction and mining and oil and gas both made negative contributions. According to the GEAD, on the other hand, manufacturing made a very large negative contribution, agriculture and forestry made a small negative contribution, and construction and mining and oil and gas made large positive contributions.

The GEAD contribution for manufacturing reflects the combination of falling real prices and falling employment that this sector was suffering during the 2000-2010 period. On the other hand, the GEAD contribution of the mining and oil and gas sector reflects the real price increases that sector enjoyed. Real price changes are also the main cause of most of the other discrepancies between the CSLS and GEAD contributions. Indeed, the theoretical prediction that the real price change terms would tend to cancel out the within-sector productivity growth terms is confirmed in Table 1: the correlation between the price contributions of the three-component GEAD (next-to-last column) and the within-sector productivity growth contributions (first column of numbers) is –0.81.

The pattern of contributions to productivity growth in Canada from the GEAD closely corresponds to contributions to nominal output growth, as shown in Table 2. Sorting the 2-digit NAICS codes based on contributions to the growth in nominal GDP results in almost the same ordering as sorting them by the contributions to productivity growth measured by the GEAD. The GEAD formula gives an exactly additive decomposition regardless of how the aggregate deflator in the formula is specified. It can therefore be used to decompose the change in nominal output per hour by letting the aggregate deflator equal 1, but the ordering of the GEAD contributions is not very sensitive to the choice of the aggregate deflator.

Exactly Additive Contributions to Productivity Growth in a Fisher Framework

The Fisher index is defined as an unweighted geometric mean of Laspeyres and Paasche indexes, but it can also be expressed as a weighted arithmetic average of these two indexes. The weights needed to express the aggregate Fisher index as an arithmetic average also provide an exactly additive decomposition. As this decomposition is not general like the GEAD, it can be termed the "Fisher exactly additive decomposition," or FEAD.

Let Q_t^L, Q_t^P and Q_t^F be the aggregate Laspeyres, Paasche and Fisher quantity indexes, respec-

Table 2

Contributions to Nominal Business Sector Output Growth and GEAD Contributions to Business Sector Labour Productivity in Canada at the Two-Digit NAICS Level, 2000-2010

(GDP at basic prices)

Sector	Nominal GDP, 2000, billions	Nominal GDP, 2010, billions	Contribution to Nominal Business Sector Output Growth (PCP)	GEAD Contribution to Annual Productivity Growth (PCP)	Relative Contribution to Nominal Output Growth	Relative Contribution to Productivity Growth
Business sector industries	777.0	1150.0	48.0	0.80	100.0	100.0
Construction	47.7	113.3	8.4	0.44	17.6	55.0
FIRE	116.5	181.8	8.4	0.20	17.5	25.0
Mining and oil and gas extraction	61.1	114.7	6.9	0.28	14.4	35.0
Professional, scientific & technical services	48.7	86.1	4.8	0.18	10.0	22.5
Retail trade	49.2	82.6	4.3	0.14	8.9	17.5
Wholesale trade	51.8	82.0	3.9	0.10	8.1	12.5
Other private services	38.7	65.8	3.5	0.12	7.3	15.0
ASWMRS	22.5	42.9	2.6	0.11	5.5	13.8
Transportation and warehousing	43.7	63.1	2.5	0.03	5.2	3.8
Information and cultural industries	31.4	49.4	2.3	0.06	4.8	7.5
Accommodation and food services	22.2	32.2	1.3	0.02	2.7	2.5
Utilities	26.3	35.3	1.2	-0.01	2.4	-1.3
Arts, entertainment and recreation	7.1	11.0	0.5	0.01	1.0	1.3
Agriculture, forestry, fishing and hunting	21.2	23.0	0.2	-0.06	0.5	-7.5
Manufacturing	188.9	166.9	-2.8	-0.83	-5.9	-103.8

Source: CSLS calculations based on Statistics Canada Canadian Productivity Accounts.

Note: Output is defined as in business sector industries as total output in all two-digit NAICS sectors except for educational services, health care and social assistance, and public administration. Owner-occupied dwellings are excluded.

Note: ASWMRS is administrative and support, waste management and remediation services; FIRE is finance and insurance and real estate and rental and leasing.

Note: PCP is short for percentage points

tively and let λ be proportional to the square root of the Paasche index. Two equivalent expressions for λ are $\lambda = (Q_t^P)^{0.5} / [(Q_t^P)^{0.5} + (Q_t^L)^{0.5}]$, and $\lambda = (P_t^P)^{0.5} / [(P_t^P)^{0.5} + (P_t^L)^{0.5}]$, where P_t^P and P_t^L are the Paasche and Laspeyres price indexes. Then:

$$Q_t^F = \frac{Q_t^L[Q_t^P]^{0.5} + Q_t^P[Q_t^L]^{0.5}}{[Q_t^P]^{0.5} + [Q_t^L]^{0.5}}$$

$$= \lambda Q_t^L + (1-\lambda)Q_t^P \qquad (13)$$

Assume, as is often the case, that the Laspeyres index is higher than the Paasche index. This implies that the weight on the Laspeyres index, denoted by λ in equation (13), is less than 0.5. For example, assuming that the indexes are not far from 100, a Laspeyres-Paasche spread of 2 percentage points would imply a λ of about 0.4975 because λ approximately equals 0.5 minus one-eighth of this spread.

A linear combination of additive contributions will itself be additive. The traditional decomposition and the CSLS decomposition are both exactly additive when Laspeyres quantity indexes are used to measure real output and also when Paasche quantity indexes are used. Therefore, an additive decomposition of the Fisher measure of productivity change can be derived as a linear combination of a decomposition of the Laspeyres measure of aggregate productivity change and a decomposition of the Paasche measure of aggregate productivity change. The weight on the Laspeyres decomposition is λ and the weight on the Paasche decomposition is $1-\lambda$.

The formulas for the CSLS decomposition in a Laspeyres volume framework are given by equations (4) and (5). As noted, the numerator of Z_{it} in that framework can be calculated by deflating nominal output in period t by Paasche price indexes, P^P_{it}.

To calculate the Paasche volume measure for the arbitrary industry i, the industry's nominal output in period 0 is multiplied by its Laspeyres price index P^L_{it}. To be able to combine the Paasche contributions with the Laspeyres ones, the Paasche productivity measures of period 0 need to have the same weighted average as the Laspeyres ones. This is accomplished by dividing all the volume measures of period 0 and period t (or all the productivity measures) by the aggregate Laspeyres price index, P^L_t. Let V_{i0} be the nominal output of industry i in time 0 and let the Paasche volume measures of productivity in industry i be denoted by z_{i0} and z_{it}. Then:

$$z_{i0} = V_{i0}\,(P^L_{it}/P^L_t)/L_{i0} \qquad (14)$$

and

$$z_{it} = V_{it}\,(1/P^L_t)/L_{i0}. \qquad (15)$$

The z_{i0} have a weighted average of Z_0 (aggregate labour productivity measured at the prices of period 0) by construction:

$$Z_0 = \Sigma_i\, l_{i0} z_{i0}. \qquad (16)$$

The aggregate Paasche productivity measure for period t, denoted by z_t, is:

$$\begin{aligned} z_t &= (\Sigma_i V_{it})(1/P^L_t)/L_t \\ &= \Sigma_i\, l_{it} z_{it} \qquad (17) \end{aligned}$$

Equation (14) implies that $l_{i0}z_{i0}/Z_0 = w_{i0}P^L_{it}/P^L_t$. Substituting $l_{i0}P^L_{it}/P^L_t$ for $l_{i0}z_{i0}/Z_0$ and writing the industry and aggregate Paasche output growth rates as $g(z_i)$ and $g(z)$, respectively, gives a convenient decomposition of the Paasche measure of productivity growth:

$$\begin{aligned} (z_t - Z_0)/Z_0 &= \Sigma_i\, [l_{i0}(z_{it} - z_{i0}) + z_{i0}(l_{it} - l_{i0}) \\ &\quad + (z_{it} - z_{i0})(l_{it} - l_{i0})]/Z_0 \\ &= \Sigma_i\, [(l_{i0}z_{i0}/Z_0)(z_{it}/z_{i0} - 1) + (z_{i0}/Z_0) \\ &\quad (l_{it} - l_{i0}) + ((z_{it} - z_{i0})/Z_0)(l_{it} - l_{i0})] \\ &= \Sigma_i\, w_{i0}(P^L_{it}/P^L_t)g(z_i) + [(z_{i0} - Z_0)/Z_0 \\ &\quad + ((z_{it} - z_{i0})-(z_t - Z_0))/Z_0](l_{it} - l_{i0}) \\ &= \Sigma_i\, w_{i0}(P^L_{it}/P^L_t)g(z_i) + [w_{i0}P^L_{it}/P^L_t - l_{i0} \\ &\quad + w_{i0}(P^L_{it}/P^L_t)g(z_i) \\ &\quad - l_{i0}g(z)](l_{it}/l_{i0} - 1) \qquad (18) \end{aligned}$$

Combining the Laspeyres productivity term for the contribution from the within-industry productivity growth in equation (4) with its Paasche counterpart in equation (18) gives the contribution of within-industry productivity growth in industry *i* to Fisher aggregate productivity growth:

$$\tilde{c}^D_i = w_{i0}\,[\lambda\, g(Z_i) + (1-\lambda)(P^L_{it}/P^L_t)g(z_i)] \qquad (19)$$

Also, the contribution of the reallocation effect to the Fisher measure of productivity change is:

$$\begin{aligned} \tilde{c}^R_i = (l_{it}/l_{i0} - 1)\{&w_{i0}[\lambda\, g(Z_i) + \\ &(1-\lambda)(P^L_{it}/P^L_t)g(z_i)] \\ &- l_{i0} + \lambda\,[w_{i0}g(Z_i) - l_{i0}g(Z)] \\ &+ (1-\lambda)[w_{i0}(P^L_{it}/P^L_t)g(z_i) \\ &- l_{i0}g(z)]\} \qquad (20) \end{aligned}$$

A Symmetrically Weighted Reallocation Effect

Following Nordhaus (2002), an economic interpretation as a measure of the Baumol effect is often assigned to the dynamic reallocation term of the contribution formulas discussed thus far.[14] Yet the dividing line between the contributions from the dynamic reallocation and within-industry productivity growth is arbitrary within a range. For any γ in the closed interval [0,1]:

$$\frac{l_{it}Z_{it} - l_{i0}Z_{i0}}{Z_0} = [(1-\gamma)l_{i0} + \gamma l_{it}]\frac{Z_{it} - Z_{i0}}{Z_0} + \frac{(1-\gamma)Z_{it} + \gamma Z_{i0}}{Z_0}(l_{it} - l_{i0}) \quad (21)$$

If $l_{it} > l_{i0}$ and $Z_{it} > Z_{i0}$, making γ larger has the effect of shifting weight to the within-industry productivity growth effect and away from the overall reallocation effect.

The traditional, CSLS and standard GEAD formulas implicitly choose a value of 0 for γ, so the dynamic reallocation term must be such that the total of the static and dynamic reallocations terms is $(Z_{it}/Z_0)(l_{it} - l_{i0})$. The implicit assumption in this approach is that the within-industry productivity growth happens before the labour reallocation process begins. A more reasonable assumption is, however, that the processes of within-industry productivity growth and labour reallocation occur simultaneously. That assumption implies a choice of 0.5 for γ. Diewert (2013), for example, achieves symmetry across the terms of his three-component GEAD by letting γ equal 0.5.

Another alternative for choosing a value for γ would be to require that a generalized change in productivity that affects all industries has no effect on the way that the reallocation effect is distributed over industries. In other words, if there is a common factor that affects every industry's productivity by the same amount and also a set of industry-specific factors, the contributions to the reallocation effect should depend only on the industry-specific factors. This invariance property can be achieved by setting γ equal to:

$$\gamma = \frac{Z_t/Z_0}{1 + Z_t/Z_0} \quad (22)$$

We can expect that most of the time aggregate productivity growth will not be great enough to make z_t/z_0 much greater than 1, so this approach to defining γ will lead to a number near 0.5.

If it is assumed that γ is equal to 0.5, $\bar{l}_i \equiv (l_{it} + l_{i0})/2$ and $\bar{Z}_i \equiv (Z_{it} + Z_{i0})/2$, then:

$$(Z_t - Z_0)/Z_0 = \Sigma_i [l_{it}Z_{it} - l_{i0}Z_{i0}]/Z_0$$

$$= \Sigma_i [\bar{l}_i Z_{i0}[(Z_{it} - Z_{i0})/Z_{i0}] + \bar{Z}_i(l_{it} - l_{i0})]/Z_0$$

$$= \Sigma_i\, 0.5[w_{i0}(1 + l_{it}/l_{i0})g(Z_i) + ((Z_{it}/Z_0) + Z_{i0}/Z_0)(l_{it} - l_{i0})] \quad (23)$$

The contribution of within-industry productivity growth for industry *i* in equation (23) is:

$$\hat{c}_i^{D*} = 0.5 w_{i0}(1 + l_{it}/l_{i0})g(Z_i) \quad (24)$$

The reallocation term in equation (23) can be written using deviations from means. This gives the symmetrically weighted version of the CSLS reallocation effect:

$$\hat{c}_i^{R*} = (l_{it} - l_{i0})0.5[(Z_{it} - Z_t)/Z_0 + (Z_{i0} - Z_0)/Z_0]$$

$$= (l_{it}/l_{i0} - 1)0.5[w_{i0}(1 + Z_{it}/Z_{i0}) - l_{i0}(1 + Z_t/Z_0)] \quad (25)$$

The formulas in equations (24) and (25) are from the Laspeyres volume framework, and

14 Baumol hypothesized that over time an increasing share of expenditures would go to products with stagnant productivity. As a result, in the long run, aggregate productivity growth would experience a slowdown. This effect came to be known as "Baumol's disease."

they have counterparts from the Paasche volume framework. The contributions to the productivity change measured using Paasche volumes are:

$$(z_t - Z_0)/Z_0 = \Sigma_i [\,\bar{l}_i\, z_{i0}[(z_{it} - z_{i0})/z_{i0}] + \bar{z}_i - 0.5(z_t + Z_0)](l_{it} - l_{i0})]/Z_0$$

$$= \Sigma_i\, 0.5[w_{i0}(P^L_{it}/P^L_t)(1+l_{it}/l_{i0})g(z_i) + (l_{it} - l_{i0})[z_{it} + z_{i0} - (z_t + Z_0)]/Z_0$$

$$= \Sigma_i\, 0.5[w_{i0}(P^L_{it}/P^L_t)(1+l_{it}/l_{i0})g(z_i) + (l_{it}/l_{i0} - 1)w_{i0}(P^L_{it}/P^L_t)[(1+z_{it}/z_{i0}) - l_{i0}(1+z_t/Z_0)] \qquad (26)$$

Combining the Laspeyres and Paasche contribution formulas gives the symmetrically weighted Fisher contribution to the growth formula. The direct contribution to aggregate Fisher productivity growth of industry i's productivity growth is:

$$\dot{c}^D_i = w_{i0}[0.5(1+ l_{it}/l_{i0})][\lambda\, g(Z_i) + (1- \lambda)(P^L_{it}/P^L_t)g(z_i)] \qquad (27)$$

The corresponding Fisher reallocation effect is:

$$\dot{c}^R_i = (l_{it}/l_{i0}-1)0.5[\lambda\, [w_{i0}(1+ Z_{it}/Z_{i0}) - l_{i0}(1+ Z_t/Z_0)] + (1-\lambda)[w_{i0}(P^L_{it}/P^L_t)(1+z_{it}/z_{i0}) - l_{i0}(1+z_t/Z_0)] \qquad (28)$$

Simplified Formula for Fisher Contributions

The Fisher contribution formula can be simplified with only a tiny loss of accuracy by assuming that at the level of detailed industries, the Laspeyres and Paasche indexes are identical. This simplification allows contributions to be calculated from the published data on Fisher indexes: the annual Laspeyres and Paasche indexes needed for the exact formula are not published by the statistical agencies. In the case of the within-industry productivity growth term, the simplified Fisher contribution formula turns out to resemble the three-component GEAD formula.

The first step in calculating simplified Fisher contributions is to determine a value for λ as a function of Laspeyres and Paasche indexes for the economy as a whole. The Laspeyres quantity index for the whole economy can be approximated as an arithmetic average of the quantity indexes for the individual industries in which the weight for any industry i is w_{i0}, industry i's share of aggregate nominal output in the base period. Similarly, the top-level Paasche index is approximated as the weighted harmonic mean of the quantity indexes for the individual industries, where the weight for any industry i is w_{it}.[15] Using these top-level Laspeyres and Paasche indexes, $\lambda = (Q^P_t)^{0.5}/[(Q^P_t)^{0.5} + (Q^L_t)^{0.5}]$.

The assumption that the Laspeyres and Paasche indexes are identical at the level of a detailed industry also implies that $g(Z_i) = g(z_i)=g(X_i)$, where $g(X_i)$ is the Fisher measure of productivity growth. Substituting $g(X_i)$ for $g(z_i)$ and $g(Z_i)$ in equation (27) implies that the within-industry productivity growth term equals $w_{i0}[0.5(1+l_{it}/l_{i0})][\lambda+(1-\lambda)(P^L_{it}/P^L_t)]g(X_i)$. But the assumption is that at the industry level the Laspeyres index equals the Fisher index, so we can drop the L superscript on P^L_{it} and write the simplified Fisher contribution of within-industry productivity growth in industry i as:

$$\dot{c}^D_i = w_{i0}[\lambda+(1-\lambda)(P_{it}/P^L_t)][0.5(1+ l_{it}/l_{i0})]\,g(X_i) \approx \dot{c}^D_i \qquad (29)$$

In equation (10), $g(p_i)$ is the growth rate of the price index of industry i relative to the

15 A harmonic mean is calculated by taking reciprocals, averaging, and then taking the reciprocal of the result.

aggregate Fisher price index, so if we approximate the aggregate Laspeyres price index P_t^L by the Fisher index, then $P_{it}/P_t^L \approx 1 + g(p_i)$. Making this substitution, equation (29) becomes:

$$\dot{c}_i^D \approx w_{i0}\,[1 + (1\text{-}\lambda)g(p_i)][1 + 0.5g(l_i)]g(X_i)$$
$$= w_{i0}g(X_i)[1 + 0.5g(l_i) + (1\text{-}\lambda)g(p_i) + 0.5(1\text{-}\lambda)g(p_i)]$$
$$\approx w_{i0}\,g(X_i)[1 + 0.5g(l_i) + 0.5g(p_i) + 0.25g(p_i)g(l_i)] \qquad (29')$$

The contribution of within-industry productivity growth in the simplified Fisher decomposition has a notable resemblance to the within-industry productivity growth in the three-component GEAD, $c_i(\Delta X_i)$ of equation (10). On the other hand, the term for the labour reallocation effect in the simplified Fisher decomposition is quite different from the one in the three-component GEAD (equation (12)). The simplified reallocation effect is:

$$\dot{c}_i^R = 0.5\{w_{i0}[\lambda + (1\text{-}\lambda)(P_{it}/P_t^L)](1+X_{it}/X_{i0}) - l_{i0}(1+X_t/X_0)]\}(l_{it}/l_{i0} - 1) \approx \grave{c}_i^R \qquad (30)$$

This reallocation effect can be written in the notation of equation (12) as:

$$\grave{c}_i^R = g(l_i)\{w_{i0}[1+(1\text{-}\lambda)g(p_i))(1+0.5g(X_i)] - l_{i0}(1+0.5g(X))\}$$

$$= g(l_i)\{w_{i0}[1+(1\text{-}\lambda)g(p_i)+0.5g(X_i) + 0.5(1\text{-}\lambda)g(p_i)g(X_i)] - l_{i0}(1+0.5g(X))\} \qquad (30')$$

Chained Fisher Volume Measures of Productivity Change

In addition to bilateral Fisher measures of productivity change that compare two years directly, the GEAD is flexible enough to be applied to chained measures of productivity change. This applicability to chained indexes is a very convenient property. Nevertheless, an algorithm for calculating additive decompositions of a chained productivity measure based on a decomposition formula that is additive in a bilateral context can easily be derived.

The notation for expressing the formula for contributions to a chained measure of productivity growth requires a time subscript t on the contribution to the change from year t–1 to year t. Also we now let Q_t^F denote the direct Fisher quantity index from year t–1 to year t and let X_t denote the index of aggregate labour productivity from year t–1 to year t. The change in aggregate productivity from year t–1 to year t is, then:

$$[Q_t^F V_{t-1}/L_t]/[V_{t-1}/L_{t-1}] - 1 = Q_t^F/(L_t/L_{t-1}) - 1$$
$$= X_t - 1$$
$$= \Sigma_i\ \grave{c}_{it}^D + \grave{c}_{it}^R \qquad (31)$$

The chained Fisher measure of aggregate productivity change from year t–1 to year t+1, equal to $X_t X_{t+1}$, then has a change of:

$$Q_{t+1}^F\ Q_t^F/(L_{t+1}/L_{t-1}) - 1 = [Q_t^F/(L_t/L_{t-1}) - 1] + [Q_t^F/(L_t/L_{t-1})][Q_{t+1}^F/(L_{t+1}/L_t) - 1]$$
$$= X_t - 1 + X_t(X_{t+1} - 1)$$
$$= \Sigma_i\ \grave{c}_{it}^D + \grave{c}_{it}^R + X_t(\grave{c}_{i,t+1}^D + \grave{c}_{i,t+1}^R) \qquad (32)$$

To add another link to the chain, we can calculate the contributions to the cumulative change in aggregate productivity as:

$$X_t - 1 + X_t(X_{t+1} - 1) + X_tX_{t+1}(X_{t+2} - 1)$$
$$= \Sigma_i\ \grave{c}_{it}^D + \grave{c}_{it}^R + X_t(\grave{c}_{i,t+1}^D + \grave{c}_{i,t+1}^R) + X_tX_{t+1}(\grave{c}_{i,t+2}^D + \grave{c}_{i,t+2}^R) \qquad (33)$$

Table 3

Exact and Approximate Estimates of Aggregate Growth of Labour Productivity in the United States, 1998-2012

(based on total economy output per full-time equivalent worker, per cent per year)

	1999	2000	2001	2002	2003	2004	2005
Using rounded exact contributions	2.51	1.81	0.69	2.79	2.96	2.54	1.38
Using approximation for exact contributions	2.48	1.73	0.61	2.80	2.96	2.49	1.40
Memo: Sum of GEAD contributions	2.48	1.74	0.63	2.79	2.98	2.46	1.40
	2006	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**
Using rounded exact contributions	0.85	0.32	-0.13	2.56	3.34	0.31	0.56
Using approximation for exact contributions	0.86	0.32	-0.12	2.52	3.39	0.31	0.59
Memo: Sum of GEAD contributions	0.85	0.33	-0.15	2.55	3.42	0.28	0.61

Source: Author's calculations based on BEA data.

Additive contributions to chained volume measures can then be calculated by rescaling the contributions to year-over-year productivity change so that they have a common base in the initial time period, and then subsequently summing over time.

Illustration using Data from the US Industry Accounts

To provide empirical evidence on the accuracy of the approximation offered by the simplified Fisher decomposition formulas of equations (27) and (28), the Annual Industry Accounts (AIAs) of the US Bureau of Economic Analysis were used to calculate simplified Fisher contributions shown in Table 3. Aggregate productivity was measured as the ratio of the production approach measure of real GDP from the AIAs to aggregate full-time equivalent employment (FTE).[16] The data covered the years 1998-2012.

In Table 3 the sum of the simplified contributions in each year is shown underneath the aggregate measure of productivity change. The residuals from subtracting the sum of the contributions from the aggregate change are small, implying that the simplified Fisher contribution formula is extremely close to being additive. The average of the residuals is 0.01 percentage points, the square root of the mean of the squared residuals is 0.038 percentage points, and most years have a residual under 0.05 percentage points in magnitude. These residuals may even understate the accuracy of the simplified Fisher formula because rounding errors in the data used for the calculations may contribute to the residuals.

To illustrate the calculation of chained measures of productivity change as described earlier, annual contributions to change for 1998-1999 up to 2011-2012 from both the simplified Fisher contribution formula and the GEAD were chained. The resulting cumulative measures of productivity change in percentage points over a 14-year interval are shown in Table 4. The real estate industry is included even though labour productivity is not a meaningful statistic for that industry because the purpose of the table is to test the performance of contributions formulas under diverse conditions. The calculated contribution of the real estate industry to labour productivity growth is large even though this

16 The production approach measure of real GDP growth is calculated by summing the exactly additive contributions to real GDP growth published in the AIAs. Labour inputs are measured using FTEs rather than hours because the labour data in the AIAs are for FTEs.

Table 4

Industry Contributions to Labour Productivity Growth in the United States Based on FEAD and GEAD Decompositions, 1998-2012

(percentage points)

Sector or Industry	Total Contribution		Within-Sector Productivity Change Contribution			Reallocation Effect Contribution		
	FEAD[b]	GEAD	FEAD[b]	CSLS-Fisher[a]	GEAD	FEAD[b]	CSLS-Fisher[a]	GEAD
Total economy	24.71	24.74	26.45	26.82	28.42	-1.75	-2.11	-3.69
Farms, forestry, fishing	0.48	0.44	0.54	0.56	0.62	-0.06	-0.07	-0.18
Oil and gas extraction	0.68	1.64	0.05	0.04	0.40	0.63	0.64	1.23
Other mining	0.26	0.77	0.19	0.19	0.19	0.06	0.07	0.58
Utilities	0.28	0.31	0.48	0.49	0.55	-0.20	-0.21	-0.24
Construction	-0.63	0.28	-0.81	-0.78	-0.75	0.18	0.15	1.02
Durable goods manufacturing excluding computers	3.13	-1.03	2.83	2.82	2.99	0.30	0.31	-4.01
Computer and electronic products	3.97	-0.18	4.30	4.40	4.74	-0.33	-0.43	-4.91
Nondurable Manufacturing	2.00	1.03	1.93	1.97	2.04	0.07	0.03	-1.01
Wholesale & retail trade	2.58	1.30	2.58	2.57	2.64	0.01	0.01	-1.34
Transportation and warehousing	0.47	0.56	0.55	0.56	0.64	-0.08	-0.10	-0.07
Publishing and motion picture and sound recording	1.11	0.61	1.27	1.29	1.29	-0.16	-0.18	-0.69
Broadcasting, data processing, telecomm. and internet	2.78	0.57	3.37	3.46	3.53	-0.59	-0.68	-2.96
Finance	2.72	1.24	2.61	2.62	2.83	0.11	0.11	-1.58
Real estate, rental and leasing	4.35	4.17	3.86	3.88	3.90	0.50	0.48	0.27
Professional, scientific, and technical services	1.26	3.55	1.03	1.04	1.07	0.22	0.22	2.48
ASWMRS	1.00	1.21	1.08	1.10	1.10	-0.08	-0.10	0.11
Educational services	-0.43	0.58	-0.03	-0.03	-0.03	-0.40	-0.40	0.61
Health care and social assistance	-0.96	3.04	0.24	0.24	0.25	-1.21	-1.21	2.80
Arts, entertainment, recreation, accommodation, food services	-0.62	1.04	0.17	0.19	0.20	-0.79	-0.81	0.85
Other services, except government	-0.57	-0.03	-0.60	-0.60	-0.59	0.03	0.03	0.56
Government	0.85	3.64	0.81	0.82	0.83	0.04	0.03	2.81

a. The CSLS-Fisher has static and dynamic reallocation effect terms, as in equation (20).

b. FEAD indicates a symmetric Fisher Exactly Additive Decomposition calculated as in equations (27) and (28).

Source: Table 3.

industry employs very little labour because the weights in the main part of the contribution formula are based on industry output, not labour inputs.

In the case of the GEAD, price changes and employment changes often influence the contribution estimates in ways that generate counter-intuitive results. In Table 4 the GEAD formula gives large negative reallocation effects for manufacturing, particularly computer manufacturing, and also for the high tech services industries of broadcasting, data processing, telecommunications and internet. Even the total contribution to US productivity growth is negative in the cases of computers and other durable goods manufacturing, which is in sharp contrast to the key role as a driver of US productivity growth that is usually assigned to these industries. In the cases of health care and oil and gas extraction, large output price increases lead to notable positive contributions to US productivity growth in the GEAD framework. Furthermore, the rapid growth of "McJobs", which replaced many of the lost jobs in US manufacturing, helped to give the sector that contains food services a significant positive contribution to US productivity growth based on the GEAD.

The simplified Fisher decomposition formula paints a very different picture from the GEAD of the sources of productivity growth. In particular, the simplified Fisher formula identifies durable goods manufacturing, computer manufacturing and high tech services as important positive drivers of US labour productivity growth. It also implies that food services and health care made negative contributions to aggregate productivity growth.

Finally, the estimates of the reallocation effect using the symmetrically weighted Fisher formula (γ=0.5) and using a formula similar to that of the CSLS for the reallocation effect (γ=0) are shown in the first two columns in the reallocation effect panel of Table 4. The two sets of estimates of contributions from labour reallocation are not very different. The negative reallocation effects are, however, a bit closer to zero using the symmetrically weighted Fisher decomposition, and for the industries with positive contributions of within-sector productivity growth, these positive contribution also tend to be closer to zero. As a result, both the reallocation effect terms and the within-industry productivity growth terms sum up to numbers that are smaller in magnitude when the symmetrically weighted formula is used.

Conclusion

The GEAD formula is widely used to decompose chained Fisher measures of productivity change because it has the advantages of yielding contributions that sum exactly to the change in aggregate productivity and it is relatively straightforward to calculate. However, the GEAD includes the direct effects of output price increases (relative to the GDP deflator) in an industry's total contribution to aggregate productivity growth, and it treats above-average increases in labour inputs as always having a positive impact on aggregate productivity. It thus tends to yield total contributions to productivity growth that resemble contributions to the growth of a rescaled version of nominal GDP per hour worked. Calculations using data from the Canada and the United States provide examples of how this type of contribution can portray the roles of industries with rapidly changing prices or labour inputs in driving aggregate productivity growth in anomalous ways.

The article also develops new decomposition formulas for measures of labour productivity based on direct Fisher and chained Fisher indexes. The simplified Fisher formula is easy to implement using the published data and produces estimates of sector and industry contributions to aggregate productive growth with useful economic interpretations. Under this approach, an industry's total contribution to aggregate productivity growth will also have the appealing property that the deviation from the average contribution (adjusted for the size of the industry) is consistent in sign with the effect that excluding that industry from the group of industries for which aggregate productivity growth is calculated would have on the measure of aggregate productivity growth.

References

Basu, Susanto and John Fernald (1995) "Aggregate Productivity and the Productivity of Aggregates," International Finance Discussion Paper 532, U.S.)) Board of Governors of the Federal Reserve System.

Bennet, T.L. (1920) "The Theory of Measurement of Changes in Cost of Living," *Journal of the Royal Statistics Society*, Vol. 83, pp. 455-462.

de Avillez, R. (2012) "Sectoral Contributions to Labour Productivity Growth in Canada: Does the Choice of Decomposition Formula Matter?" *International Productivity Monitor*, Number 24, Fall, pp. 97-117.

Diewert, W. Erwin (2013) "Decompositions of Productivity Growth into Sectoral Effects", paper presented at IARIW-UNSW Conference on Productivity: Measurement, Drivers and Trends, Sydney, Australia, November 26-27.

Diewert, W.E. (1978) "Superlative Index Numbers and Consistency in Aggregation," *Econometrica*, Vol. 46, pp. 883-900.

Dumagan, Jesus C. (2013) "A Generalized Exactly Additive Decomposition of Aggregate Labour Productivity Growth," *Review of Income and Wealth*, Vol. 59, No. 1, March, pp. 157-168.

Frisch, Ragnar (1936) "The Problem of Index Numbers," *Econometrica*, Vol. 4, pp. 1-38.

Hulten, Charles (1978) "Growth Accounting With Intermediate Inputs," *Review of Economic Studies*, Vol. 45, pp. 511-518.

Lewrick, Ulf, Lukas Mohler and Rolf Weder (2014) "When Firms and Industries Matter: Trade and Productivity Growth," presented at the 48th Annual Conference of the Canadian Economics Association, May 30-July 1, 2014, Simon Fraser University, Vancouver, B.C.

Moyer, Brian, Marshall Reinsdorf and Robert Yuskavage (2004) "Aggregation Issues in Integrating and Accelerating BEA's Accounts: Improved Methods for Calculating GDP by Industry" in *A New Architecture for the U.S. National Accounts*, D.W. Jorgenson, J.S. Landefeld and W.D. Nordhaus, eds., University of Chicago Press for the Conference on Research in Income and Wealth (CRIW).

Nordhaus, William D. (2002) "Productivity Growth and the New Economy," *Brookings Papers on Economic Activity*, Vol. 2, pp. 211-264.

Oliner, Stephen D., and Daniel E. Sichel (2000) "The Resurgence of Growth in the Late 1990s: Is Information Technology the Story?" *Journal of Economic Perspectives*, Vol. 14, Fall, pp. 3-22.

Pollak, Robert A. (1981) "The Social Cost-of-Living Index" *Journal of Public Economics*, Vol. 15, June, 311-36. Reprinted in Pollak, Robert A. (1989) The Theory of the Cost-of-Living Index Oxford Univ. Press, New York, pp. 128-152.

Olley, G. S., and A. Pakes (1996) "The Dynamics of Productivity in the Telecommunications Equipment Industry," *Econometrica*, Vol. 64 (6), pp. 1263-1297.

Reinsdorf, Marshall B., W. Erwin Diewert, and Christian Ehemann (2002) "Additive Decompositions for Fisher, Törnqvist and Geometric Mean Indexes," *Journal of Economic and Social Measurement*, Vol. 28, pp. 51–61.

Reinsdorf, Marshall B. (2010) "Terms of Trade Effects: Theory and Measurement." *Review of Income and Wealth*, Vol. 56, June, pp. S177-S205.

Reinsdorf, Marshall and Robert Yuskavage (2010) "Exact Industry Contributions to Labour Productivity Change," in W.E. Diewert, B.M. Balk, D. Fixler, K.J. Fox and A.O. Nakamura, *Price and Productivity Measurement: Volume 6 - Index Number Theory*, pp. 77-102. Trafford Press.

Sharpe, Andrew (2010a) "The Paradox of Market-Oriented Public Policy and Poor Productivity Growth in Canada," *A Festschrift in Honour of David Dodge.* Ottawa: Bank of Canada. http://www.bankofcanada.ca/wp-content/uploads/2010/09/sharpe.pdf

Sharpe, Andrew (2010b) "Can Sectoral Reallocations of Labour Explain Canada's Abysmal Productivity Performance," *International Productivity Monitor*, No. 19, Spring, pp. 40-49.

Stiroh, Kevin J. (2002) "Information Technology and the US Productivity Revival: What Do the Industry Data Say?" *American Economic Review*, Vol. 92, December, pp. 1559-1576.

Tang, J. and W. Wang (2004) "Sources of Aggregate Labour Productivity Growth in Canada and the United States," *Canadian Journal of Economics*, Vol. 37, pp. 421-444.

Appendix 1

Industry Contributions to Total Factor Productivity in a Growth Accounting Framework

Under certain assumptions, aggregate TFP growth can be expressed as a weighted sum of sector contributions using the weights that were introduced by Domar (1961). This result is noteworthy because usually changes in aggregate productivity cannot be completely explained by patterns of within-industry productivity growth.

There are many ways to measure the relative distance between production possibility frontiers (PPFs) attributable to growth of TFP, but two of them are especially relevant. Let Y^F be the vector of final outputs, M^M be the vector of imported intermediate inputs, and let L_t, K_t and N_t be the economy's endowments of primary factors of labour, capital and natural resources (land) at time t.[17] For purposes of exposition, it is convenient to assume that the economy is at a profit-maximizing point on its production possibility frontier, which rules out most kinds of disequilibria, and that aggregate technology exhibits constant returns to scale. Define the revenue function $R_t(L,K,N;P^F, P^M)$ as the function that gives the maximum value of revenue $P^F \cdot Y^F - P^M \cdot M^M$ achievable at prices (P^F, P^M) with the technology of period t and primary factor inputs (L,K,N). Then a measure of aggregate TFP based on final period prices and final period inputs is:

$$TFP^{Allen\text{-}Paashe} = R_t(L_t,K_t,N_t;P^F_t,P^M_t) / R_0(L_t,K_t,N_t;P^F_t,P^M_t)$$

$$= [R_t(L_t,K_t,N_t;P^F_t,P^M_t) / R_0(L_0,K_0,N_0;P^F_t,P^M_t)] / [R_0(L_t,K_t,N_t;P^F_t,P^M_t) / R_0(L_0,K_0,N_0;P^F_t,P^M_t)] \quad \text{(A1)}$$

The Paasche quantity index of GDP provides an upper bound estimate of the total change in output:[18]

$$(P^F_t \cdot Y^F_t - P^M_t \cdot M^M_t)/(P^F_t \cdot Y^F_0 - P^M_t \cdot M^M_0) \approx R_t(L_t,K_t,N_t;P^F_t,P^M_t) / R_0(L_0,K_0,N_0;P^F_t,P^M_t) \quad \text{(A2)}$$

If technology change has the same proportional effect on output when inputs are (L_t,K_t,N_t) as when they are (L_0,K_0,N_0) then:

$$R_t(L_t,K_t,N_t;P^F_t,\ P^M_t)/R_t(L_0,K_0,N_0;P^F_t,\ P^M_t) = R_0(L_t,K_t,N_t;P^F_t,P^M_t)/R_0(L_0,K_0,N_0;P^F_t,P^M_t) \quad \text{(A3)}$$

Furthermore, if factors of production are paid their marginal revenue product, a Paasche quantity index of inputs will provide a lower bound approximation to $R_t(L_t,K_t,N_t;P^F_t,P^M_t)/R_t(L_0,K_0,N_0;P^F_t,P^M_t)$. The Paasche quantity index of output divided by the Paasche quantity index of inputs is then an upper bound measure of the theoretical change in total factor productivity given by $TFP^{Allen\text{-}Laspeyres}$.

A symmetric analysis shows that under certain assumptions a Laspeyres quantity index is:

$$TFP^{Allen\text{-}Laspeyres} = R_t(L_0,K_0,N_0;P^F_0,P^M_0) / R_0(L_0,K_0,N_0;P^F_0,P^M_0)$$

$$= [R_t(L_t,K_t,N_t;P^F_0,P^M_0) / R_0(L_0,K_0,N_0;P^F_0,P^M_0)] / [R_t(L_t,K_t,N_t;P^F_0,P^M_0) / R_t(L_0,K_0,N_0;P^F_0,P^M_0)]$$

$$= [R_t(L_t,K_t,N_t;P^F_0,P^M_0) / R_0(L_0,K_0,N_0;P^F_0,P^M_0)] / [R_0(L_t,K_t,N_t;P^F_0,P^M_0) / R_0(L_0,K_0,N_0;P^F_0,P^M_0)] \quad \text{(A4)}$$

17 The time subscript on N could reflect exhaustion or new discoveries of mineral resources, or changes in the amount of land usable for agriculture caused by global warming.

18 Although the Paasche index is a lower bound in the theory of consumption price indexes, in the theory of output price indexes it is an upper bound. The quantity mix of the initial period may not be revenue-maximizing at the prices of the final period.

The Laspeyres quantity index of output is $(P_0^F \cdot Y_t^F - P_0^M \cdot M_t^M)/(P_0^F \cdot Y_0^F - P_0^M \cdot M_0^M)$. Under certain assumptions, dividing this index by a Laspeyres quantity index for inputs gives a theoretical lower bound for the conceptual measure given by $TFP^{Allen\text{-}Laspeyres}$. The Fisher index of TFP then has an appealing property as an average of upper and lower bounds for theoretical indexes.

Aggregate and Industry Level TFP in the Framework of the Domar Decomposition

Besides the final goods and services included in Y^F, industries also produce outputs that are used as intermediate inputs by themselves or by other industries. Assuming, for simplicity, that there are no taxes on products or tariffs, nominal GDP can be calculated as the sum of the value added of every industry. The Laspeyres (Paasche) volume measure of real GDP can also be calculated as the sum of industries' value added measured at initial (final) period prices.

The assumption that the economy is operating at a profit maximizing point on the PPF implies that at the margin reallocating inputs from one industry to another will not change the value of the revenue function. Hulten (1978) showed that in this framework the log change in aggregate TFP defined as an outward shift in the PPF can be calculated as a weighted sum of the log change in TFP of industries using the weights introduced by Domar (1961). The Domar weights add up to more than 1. Define Y_{i0} as the nominal gross output of industry i excluding intermediate inputs used within industry i and define G_0 as the total value added of all industries. Then industry i's Domar weight w_i^D equals Y_{i0} divided by aggregate value added G_0.

Let π^G be the Paasche index that measures period t prices relative to period 0 prices for G. Then the change in the Laspeyres quantity index for aggregate output, denoted $g^L(G)$, is:

$$g^L(G) = \frac{G_t/\pi^G - G_0}{G_0} \qquad \text{(A5)}$$

To define the aggregate quantity index of primary inputs used in the Domar decomposition I_t we must either treat detailed inputs used by different industries as different items, or assume that detailed inputs receive the same wage (or returns) everywhere they are employed. (If inputs in different industries are treated as different items in the quantity index of aggregate inputs, when labour is reallocated from a low wage industry to a high wage industry, the weight on the increase in labour in the high wage industry will be greater than the weight on the decrease in labour in the low wage industry and the aggregate input quantity index will rise.) In addition, it is assumed that an industry's revenues from sales of output are all used to acquire intermediate inputs or pay factors of production. Thus, if J_{i0} denotes the cost of the intermediate inputs that industry i obtains from other industries plus the cost of the primary inputs employed in industry i, $J_{i0} = Y_{i0}$. Let the Laspeyres measure of the growth rate of aggregate primary inputs I be $g^L(I) = (I_t/\pi^I - I_0)/I_0$, where π^I is a Paasche price index for inputs. Then the Laspeyres quantity index measure of aggregate TFP is

$$\begin{aligned} TFP^{Laspeyres} &= \frac{g^L(G) - g^L(I)}{1 + g^L(I)} \\ &\approx g^L(G) - g^L(I) \\ &= \textstyle\sum_i w_i^D \; [g^L(Y_i) - g^L(J_i)] \\ &= \textstyle\sum_i w_i^D \; TFP_i^{Laspeyres} \qquad \text{(A6)} \end{aligned}$$

In the framework of the Domar decomposition, an industry's own TFP growth times its Domar weight gives its contribution to aggregate TFP growth.

Appendix 2
Index of Labour Inputs that uses Compensation to Weight Industry-Occupation Cells

In a competitive neo-classical equilibrium, the marginal revenue product of a labour input is equal to the amount that the employer has to pay in compensation costs (wage plus benefits and social contributions) to employ the labour. However if labour is treated as a homogeneous input, the formula for the contribution of labour reallocation to aggregate productivity growth must assume that the marginal product of labour varies in direct proportion to its average product as measured by the ratio of real value added to the quantity of labour inputs used. Furthermore, differences in pay levels across industry-occupation cells may reflect differences in training, aptitude and experience. If so, industry-occupation cells should be treated as different kinds of inputs. When this is done, the role of reallocation effects (which are a kind of residual that cannot be explained by within-industry productivity growth) may be reduced.

To calculate a Laspeyres quantity index for labour inputs, let B_{it} be the nominal wage bill in year t (for convenience, we use "wages" as equivalent to compensation costs). Also, let W_t^P be the aggregate Paasche price index for wages and let W_{it}^P be the Paasche index of wages. The Laspeyres volume of labour inputs at time t is, then,

$$\hat{B}_t = \sum_i B_{it}/W_{it}^P$$

$$= \sum_i \hat{B}_{it}$$

$$= B_t / W_t^P \quad \text{(A7)}$$

Let $\hat{b}_{it} = b_{it}(W_t^P / W_{it}^P) = (B_{it} / W_{it}^P)/(B_t / W_t^P)$, the share of the aggregate wage bill paid by industry i if the wage rates of year 0 had prevailed in year t, and let $b_{i0} = B_{i0} / B_0$, the industry i's share of the aggregate wage bill in year 0. Also, let $\hat{Z}_{it} = (V_{it}/ P_{it}^P)/(B_{it}/ W_{it}^P)$, the Laspeyres volume measure of labour input productivity. Letting $V_t = \sum_i V_{it}$ be nominal GDP at time t and $\hat{V}_t = V_t/ P_t^P$, the change in the aggregate measure of Laspeyres labour input productivity is:

$$\frac{\hat{V}_t/\hat{B}_t}{V_0/B_0} - 1 = (\hat{Z}_t - \hat{Z}_0)/\hat{Z}_0$$

$$= \sum_i [\hat{b}_{it}\ \hat{Z}_{it} - b_{i0}\hat{Z}_{i0}]/\hat{Z}_0$$

$$= \sum_i [0.5(b_{i0} + \hat{b}_{it})(\hat{Z}_{it} - \hat{Z}_{i0}) + 0.5(\hat{Z}_{i0} + \hat{Z}_{it})(\hat{b}_{it} - b_{i0})]/\hat{Z}_0$$

$$= \sum_i 0.5(1+\hat{b}_{it}/b_{i0})g(\hat{Z}_i) + \sum_i [0.5(\hat{Z}_{i0} + \hat{Z}_{it}) / \hat{Z}_0](\hat{b}_{it} - b_{i0})$$

$$= \sum_i 0.5(1+\hat{b}_{it}/b_{i0})g(\hat{Z}_i) + \sum_i [0.5(\hat{Z}_{i0} + \hat{Z}_{it} - (\hat{Z}_0 + \hat{Z}_t))/\hat{Z}_0](\hat{b}_{it} - b_{i0}) \quad \text{(A8)}$$

The contribution to aggregate Laspeyres labour input productivity growth from within-industry labour input productivity growth in industry i is:

$$c_i^{L-D} = 0.5(1+\hat{b}_{it}/b_{i0})g(\hat{Z}_i) \quad \text{(A9)}$$

The contribution of reallocation of labour inputs to or from industry i to aggregate Laspeyres labour input productivity growth is therefore:

$$c_i^{L-R} = [0.5(\hat{Z}_{i0} + \hat{Z}_{it} - (\hat{Z}_0 + \hat{Z}_t))/\hat{Z}_0](\hat{b}_{it} - \mathrm{b}_{i0}) \quad \text{(A10)}$$

To derive the Paasche volume index of labour inputs, let W_{it}^L be the Laspeyres index of wages in industry i, and let W_t^L be the aggregate Laspeyres index of wages. Also, let $\hat{b}_{i0}$ be the share of the aggregate wage bill that would have been paid by industry i had the prices of period t prevailed in period 0

$$\hat{b}_{i0} = b_{i0}(W^L_{it}/W^L_t) \quad \text{(A11)}$$

Then the labour inputs productivity level of industry i in period 0 measured at prices of period t is:

$$\hat{z}_{i0} = \frac{V_{i0}(P^L_{it}/P^L_t)}{B_{i0}(W^L_{it}/W^L_t)} \quad \text{(A12)}$$

and the aggregate Paasche volume productivity equals:

$$\hat{z}_0 = V_0/B_0$$

$$= \sum_i \hat{b}_{i0}\ \hat{z}_{i0} \quad \text{(A13)}$$

Now let $\hat{z}_{it} = (V_{it}/B_{it})(W^L_t/P^L_t)$, a normalized ratio of value added to total wages in industry i, and let $\hat{z}_t = (V_t/B_t)(W^L_t/P^L_t)$ denote aggregate labour input productivity in period t. If A_0 is the aggregate ratio of output to labour inputs measured in current dollars in the base period, the aggregate Paasche volume measure of labour input productivity is:

$$\frac{(V_t/P^L_t)(B_t/W^L_t)}{V_0/B_o} - 1 = \frac{\hat{z}_t - \hat{z}_0}{\hat{z}_0}$$

$$= \frac{\Sigma_i b_{it}\hat{z}_{it} - (\hat{b}_{i0}\hat{z}_{i0})}{\hat{z}_0}$$

$$= \sum_i [0.5(\hat{b}_{i0} + b_{it})(\hat{z}_{it} - \hat{z}_{i0}) + 0.5(\hat{z}_{i0} + \hat{z}_{it})(b_{it} - \hat{b}_{i0})]/\hat{z}_0$$

$$= \sum_i 0.5(1 + b_{it}/\hat{b}_{i0})g(\hat{z}_i) + \sum_i [0.5(\hat{z}_{i0} + \hat{z}_{it})/\hat{z}_0](\hat{b}_{it} - b_{i0})$$

$$= \sum_i 0.5(1 + b_{it}/\hat{b}_{i0})g(\hat{z}_i) + \sum_i [0.5(\hat{z}_{i0} + \hat{z}_{it} - (\hat{z}_0 + \hat{z}_t))/\hat{z}_0](\hat{b}_{it} - b_{i0}) \quad \text{(A14)}$$

The contribution to aggregate Paasche labour input productivity growth from within-industry labour input productivity growth in industry i is:

$$c_i^{P-D} = 0.5(1 + b_{it}/\hat{b}_{i0})g(\hat{z}_i) \quad \text{(A15)}$$

The contribution of reallocation of labour inputs to or from industry i to aggregate Laspeyres labour input productivity growth is therefore:

$$c_i^{P-R} = [0.5(\hat{z}_{i0} + \hat{z}_{it} - (\hat{z}_0 + \hat{z}_t))/\hat{z}_0](b_{it} - \hat{b}_{i0}) \quad \text{(A16)}$$

Finally, we can use λ from equation (10) to define Fisher index contributions to aggregate labour inputs productivity change. The direct Fisher contribution of within-industry productivity growth is then seen to be:

$$c_i^{F-D} = \lambda c_i^{L-D} + (1-\lambda) c_i^{P-D}$$

$$= \lambda 0.5(1 + \hat{b}_{it}/b_{i0})g(\hat{Z}_i) + (1-\lambda)0.5(1 + b_{it}/\hat{b}_{i0})g(\hat{z}_i)$$

$$= 0.5[1 + \lambda(\hat{b}_{it}/b_{i0})g(\hat{Z}_i) + (1-\lambda)(b_{it}/\hat{b}_{i0})g(\hat{z}_i)] \quad \text{(A17)}$$

The contribution of reallocation of labour inputs involving industry i to Fisher labour inputs productivity is then:

$$c_i^{F-R} = \lambda c_i^{L-R} + (1-\lambda) c_i^{P-R}$$

$$= 0.5[\lambda[(\hat{Z}_{i0} + \hat{Z}_{it} - (\hat{Z}_0 + \hat{Z}_t))/\hat{Z}_0](\hat{b}_{it} - b_{i0}) + (1-\lambda)[(\hat{z}_{i0} + \hat{z}_{it} - (\hat{z}_0 + \hat{z}_t))/\hat{z}_0](b_{it}/\hat{b}_{i0})]] \quad \text{(A18)}$$